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FUZZY PROJECTIVITY

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Abstract: The author studied about fuzzy G-modules and fuzzy representations in[8] and fuzzy injectivity in [9]. Cartan and Eilemberg introduced the notion of projectivity of modules in [3]. As a continuation of [8] and [9], here the concept fuzzy G-module projectivity is introduced and analysed.

I.Introduction

The *fuzzy set theory* was introduced by L.A.Zadeh[1] in 1965. There were several attempts to fuzzify various mathematical structures. The fuzzification of algebraic structures was initiated by Rosenfield[2]. He introduced the notions of fuzzy subgroupoids and fuzzy subgroups and obtained their basic properties.

In the paper [6], Mac Lane formulated the projective and injective lifting properties for the category of abelian groups. Charles. A. Weibel [7] describes free and divisible abelian groups respectively. But he did not come to the notion of projective modules because he did not apply these lifting properties to categories of modules. Cartan and Eilemberg introduced the notion of projectivity of modules in [3]. The author studied about fuzzy G-modules and fuzzy representations in [8] and fuzzy injectivity in [9]. As a continuation of [8] and [9], here the concept fuzzy G-module projectivity is introduced and analysed.

2. Preliminaries

The terms and notations used in this paper are either standard or are explained as and when they first appear.

2.1 Definition. A G-module M is ***projective*** if for any G-module M* and any G-submodule N of M*, any homomorphism $\varphi : M \to M^*/N$ can be lifted to a homomorphism $\psi: M \to M^*$. That is, $\pi \circ \psi = \varphi$, where $\pi : M^* \to M^*/N$ is the natural homomorphism.

2.2 Example. Let G = C-{0} and M = C. Then M is a G-module. Since G = C-{0}, except the zero G-submodules, no proper subset of C becomes a G-module. If M* is any other G-module, then M* is anyone of the following: (i) M* = {0}; (ii) M* = C^n ($n \geq 1$) or M* is a G-submodule of C^n; (iii) M* = l^p ($1 \leq p \leq \infty$) or M* is a G-submodule of l^p; (iv) M* = Space of all functions from any set S into C; (v) M* = Space of polynomial functions over the field C; (vi) M* = Space of all m x n matrices over the field C. Let N be any G-submodule of M* and let $\varphi : M \to M^*/N$ be a homomorphism. (i) Here N = M* = {0}, and so the homomorphism $0 = \psi: M \to M^*$ lifts φ. (ii) Since C^n is n-dimensional, Dim.N = m $\leq$ Dim.M* = t $\leq$ n. Let $\{\alpha_1, \alpha_2, \ldots \alpha_m\}$ be a basis for N such that $\{\alpha_1, \alpha_2, \ldots \alpha_m \ldots \alpha_t\}$. ($t \leq n$) is a basis for M*. Then $\{\alpha_{m+1}+N, \alpha_{m+2}+N, \ldots\ldots\ldots, \alpha_t+N\}$ is a basis for M*/N. Then $\psi : M \to M^*$ defined by

$$\psi(m) = \sum_{i=m+1}^{t} c_i \alpha_i, \text{ where } \varphi(m) = \sum_{i=m+1}^{t} c_i (\alpha_i+N)$$

is a homomorphism and it lifts the homomorphism φ. For let $\pi : M^* \to M^*/N$ be the projecton map.Then

$$\begin{aligned} \pi \bullet \psi(m) &= \pi(\psi(m)) \\ &= \pi\left(\sum_{m+1}^{t} c_i \alpha_i\right) \\ &= \sum_{m+1}^{t} c_i (\alpha_i+N) \\ &= \varphi(m) \end{aligned}$$

Therefore $\pi \bullet \psi = \varphi$, so ψ lifts the homomorphism φ. (iii) follows from (ii). (iv) Here $M^* = M_1 \oplus M_2$, where M_1 is the G-submodule of M of all odd functions and M_2 is the G-submodule of M of all even functions. Then as in (ii), there exits a homomorphism $\psi : M \to M^*$, which lifts φ. (v) Follws from (iv). (vi) Since M* is an mn-dimensioal vectorspace over C, as in (ii), there exits a homomorphism $\psi : M \to M^*$, which lifts φ . Therefore in any case, the homomorphism φ can be lifted to a homomorphism $\psi : M \to M^*$. Therefore M is projective. □

2.3. Definition. Let M, M* be G-modules. Then M is ***M*- projective*** if for any G-submodule N of M*, any homomorphism $\varphi : M \to M^*/N$ can be lifted to a homomorphism $\psi : M \to M^*$.

2.4. Example. As in example 2.2, let G = C −{0} and M =C. Let $M^* = C^n$ ($n \geq 1$). Then M and M* are G-modules . Let N be any G-submodule of M*. Then from (ii) of example2.2, M is M*- projective.

2.5. Definition: A ***sequence*** of G-modules $M_1 \xrightarrow{\theta_1} M_2 \xrightarrow{\theta_2} M_3 \xrightarrow{\theta_3} \ldots\ldots$ consists of an ordered family of G-modules $M_1, M_2, M_3, \ldots\ldots\ldots$with G-module homomorphisms $\theta_1 : M_1 \to M_2$, $\theta_2 : M_2 \to M_3$, etc. For $i > 1$, we say that the sequence is ***exact at Mi*** if the image θ_{i-1} coincides with the kernel of

θ_i. That is, $\theta_{i-1}(M_{i-1}) = Ker(\theta_i)$.

2.8. Example. Let G={1,-1}.Consider the G-modules $M_1 = \{0\}$, $M_2 = C$, $M_3 = R$, $M_4 = \{0\}$ over R. Consider the sequence $M_1 \xrightarrow{\theta_1} M_2 \xrightarrow{\theta_2} M_3 \xrightarrow{\theta_3} M_4$ with homomorphisms $\theta_1(0)=0$, $\theta_2(x+iy)=x+y$, $(x+iy \in M_2)$, $\theta_3(x)=0$, $(x \in M_3)$. Then the above sequence is exact at M_3 because $\theta_2(M_2) = M_3 = Ker\ \theta_3$ □

2.7. Definition. An exact sequence $\{0\} \to M \xrightarrow{\mu} N \to \{0\}$ ***splits*** if there exits a homomorphism v: N→M such that $\mu \bullet v = 1$, the identity map on N.

2.8. Example. The sequence in the example 2.8. splits because the map v : $M_3 \to M_2$ defined by $v(x)=x+i0$ $(x \in M_3)$ is a homomorphism and $\theta_2 \bullet v = 1$, the identity map on M_3. □

2.9. Proposition. *Let M and M* be G-modules such that M is M*-projective. Then any epimorphism $f: M^* \to M$ splits.*

Proof. Let N =ker.f. Then N is a G-submodule of M* and $f: M^* \to M$ is an onto homomorphism with kernel N , and hence $M^*/N \approx M$.Let $\varphi: M \to M^*/N$ be the isomorphism. Since M is M*-projective, φ can be lifted to a homomorphism $\psi: M \to M^*$ .Therefore ∋ a homomorphism $\psi: M \to M^*$ such that $f \bullet \psi = I$, identity on M. Hence the epimorphism $f: M^* \to M$ splits. □

2.10. Proposition. *Let M and M* be G-modules such that M is M*-projective. Let N be any G-submodule of M*. Then M is N -projective and M is M*/ N -projective.*

Proof. First to prove M is N -projective.Let Y be any G-submodule of N and $\varphi: M \to N/Y$ be a homomorphism. Let $\varphi_1: N/Y \to M^*/Y$ be the inclusion homomorphism. Since M is M* -projective, ∋ a homomorphism $\psi: M \to M^*$, which lifts $\varphi \bullet \varphi_1$. ie $\pi \bullet \psi = \varphi \bullet \varphi_1$, where $\pi: M^* \to M^*/Y$ is the projection map. Let $\pi_1 = \pi|N$. Since φ_1 is the inclusion homomorphism, ∋ a homomorphism $\psi_1: M \to N$ such that $\pi_1 \bullet \psi_1 = \varphi$. Therefore M is N -projective. Now to prove M is M*/N -projective. Let Y/N be any G-submodule of M*/N and let $f: M \to (M^*/N)/(Y/N) \to M^*/Y$ be a homomorphism. Consider the projections $\pi: M^* \to M^*/Y$ and $\pi^1: M^* \to$

M^*/N . Since $f : M \to M^*/Y$ is a homomorphism and M is M^*-projective, the homomorphism f can be lifted to a homomorphism $g : M \to M^*$. ie, $\pi \bullet g = f$. Then the map $\pi' \bullet g : M \to M^*/N$ is a homomorphism and the homomorphism f can be lifted to $\pi' \bullet g$ because $\pi'' \bullet (\pi' \bullet g) = f$, where $\pi'' : M^*/N \to (M^*/N)/(Y/N)$ be the projection map. Therefore M is M^*/N - projective. □

2.11. Proposition. *A direct sum $M = \overset{n}{\underset{1}{\oplus}} M_i$ is M^* -projective if and only if M_i is M^* -projective, where M_i, M, M^* are all G-modules.*

Proof. (⇒) Assume that $M = \overset{n}{\underset{1}{\oplus}} Mi$ is M^*-projective. Let $\pi_i : M \to M_i$ ($1 \le i \le n$) be the projection map. Let N be any G-submodule of M^* and $\varphi_i : M_i \to M^*/N$ be a homomorphism. Then $\varphi_i \bullet \pi_i : M \to M^*/N$ is a homomorphism. Since M is M^*-projectivite, $\varphi_i \bullet \pi_i$ can be lifted to a homomorphism $\psi : M \to M^*$ .ie, $\pi \bullet \psi = \varphi_i \bullet \pi_i$, where $\pi : M^* \to M^*/N$ is the projection. Then $\psi_i = \psi/M_i : M_i \to M^*$ is a homomorphism and it lifts the homomorphism φ_i because $\pi \bullet \psi_i = \varphi_i$. Therefore M_i is M^*-projective.

(⇐) Suppose M_i is M^*-projective and $M = \overset{n}{\underset{1}{\oplus}} M_i$. Let N be any G-submodule of M^* and $\varphi : M \to M^*/N$ be any homomorphism. Let $\varphi_i : M_i \to M$ ($1 \le i \le n$) be the inclusion homomorphism.Then $\varphi \bullet \varphi_i : Mi \to M^*/N$ is a homomorphism. Since M_i is M^*-projective, $\varphi \bullet \varphi_i$ can be lifted to a homomorphism $\psi_i : M \to M^*$. ie, $\pi \bullet \psi_i = \varphi \bullet \varphi_i$, where $\pi : M^* \to M^*/N$ is the projection. Let $\psi_i = \psi_i/M_i$ for all i. Then $\psi : M \to M^*$ is a homomorphism and $\pi \bullet \psi = \varphi$. Therefore M is M^*-projective. □

2.12. Proposition. *Let M, M_i ($1 \le i \le n$) be G-modules. Then M is $\overset{n}{\underset{1}{\oplus}} M_i$-projective if and only if M is M_i-projective for all i.*

Proof. (⇒) This part follows from Proposition 2.12.

(⇐) Suppose M is M_i-projective, for all i and let $M^* = \overset{n}{\underset{1}{\oplus}} M_i$. To prove M is M^*-projective. Let N be any G-submodule of M^* and $\varphi : M \to M^*/N$ be any homomorphism. For $m \in M$, $\varphi(m) = \Sigma m_i' + N$; $m_i' \in M_i$. If also $\varphi(m) = \Sigma n_i' + N$; $n_i' \in M_i$. Then $\Sigma (m_i' - n_i') \in N$. Therefore $m_i' - n_i' \in N \cap M_i$

$(1 \leq i \leq n)$. In otherwords, $\varphi(m)$ determines m_i' modulo $M_i \cap N$ .So φ gives the map φ_i: $M \to M_i/(M_i \cap N)$ by $\varphi_i(m) = m_i' + (M_i \cap N)$. Since $M_i \cap N$ is a G-submodule of M_i and M is M_i-projective, for all i, φ_i $(1 \leq i \leq n)$ can be lifted to a homomorphism $\psi_i : M \to M_i$. Put all these ψ_i $(1 \leq i \leq n)$ together, we get $\psi: M \to \bigoplus_1^n M_i = M^*$, which is well defined because $\varphi(m) = \Sigma\, m_i' + N$ only finitely many m_i's are non-zero. Also the map $\psi: M \to M^*$ lifts φ because ψ_i lifts φ_i f or all i. Therefore M is $M^* = \bigoplus_1^n M_i$-projective. □

2.13. Definition. A G-module M is ***quasi-projective*** if M is M-projective.

2.14. Example. In example 2.4, if G ={ 1,-1, i , -i } and M = M* = C , then M is quasi-projective. □

2.15. Definition. Two G-modules M and M* are said to be ***relatively projective*** if M is M*-projective and M* is M-projective.

3. Projectivity and Quasi-projectivity of Fuzzy G-modules.

3.1. Definition. Let M and M* be G-modules. Let μ and ν be fuzzy G-modules on M and M* respectively. Then μ is ***v-projective*** if (i) M is M*-projective and (ii) $\mu(m) \leq \nu(\psi(m))$, for all $\psi \in$ Hom (M,M*)

3.2. Example. In example 2.4, if G ={l,i,-1,-i} we have M = C is $M^* = C^n$ -projective. Define $\mu : M \to [0, 1]$ by

$$\mu(x) = 1, \quad \text{if } x = 0$$
$$= \tfrac{1}{2}, \quad \text{if } x \neq 0$$

Then μ is a fuzzy G-module on M. Define $\nu : M^* \to [0,1]$ by

$$\nu(x) = 1, \quad \text{if } x = 0$$
$$= \tfrac{3}{4}, \quad \text{if } x = 0$$

Then ν is a fuzzy G-module on M*. Also $\mu(m) \leq \nu(\psi(m))$, for all $\psi \in$ Hom (M,M*). Therefore μ is ν - projective. □

3.3. Proposition. *Let M be a G-module and N be a G-submodule of M. If M has a fuzzy G-module, then both G-submodules N and M/N has fuzzy G-modules .*

Proof. Follows form Proposition 3.4 of [9]. □

3.4. Proposition. *Let M and M* be G-submodules such that M* is finite dimensional and M is M*-projective. Let* $B = \{\beta_1, \beta_2, \ldots\ldots \beta_n\}$ *be a basis for M*. Let* μ *and* ν *be fuzzy G-modules on M and M* respectively. If* $\mu(m) \leq \wedge_j \nu(\beta_j)$ *for all* $m \in M$*, then* μ *is* ν*-projective.*

Proof. Let $\psi \in Hom(M,M^*)$. Then for any $m \in M$, $\psi(m) \in M^*$. So $\psi(m) = c_1\beta_1 + c_2\beta_2 + \ldots\ldots + c_n\beta_n$, where c_i's are scalars.

$$\nu(\psi(m)) = \nu(c_1\beta_1 + c_2\beta_2 + \ldots\ldots + c_n\beta_n) \geq \wedge_j \nu(\beta_j) \quad (1)$$

Given that $\wedge_j \nu(\beta_j) \geq \mu(m), m \in M$. (2)

From (1) and (2), $\mu(m) \leq \nu(\psi(m)), \forall m \in M$ and $\psi \in Hom(M,M^*)$. Therefore μ is ν-projective. □

3.5. Proposition. *Let M and M* be G-modules and* μ, ν *be fuzzy G-modules on M and M* respectively such that* μ *is* ν *-projective. If N is a G-submodule of M* and* ν' *is any fuzzy G-module on N, then* μ *is* ν'*-projective if* ν' *exceeds* ν *on N.*

Proof. Given μ is ν-projective. Then (i) M is M*-projective and (ii) $\mu(m) \leq \nu(\psi(m)), \forall m \in M$ and $\psi \in Hom(M,M^*)$. Since M is M*-projective and N is a G-submodule of M*, from Proposition 2.10, M is N-projective. Let $\varphi \in Hom(M,N)$ and $\eta : N \rightarrow M^*$ be the inclusion homomorphism. Then $\eta \bullet \varphi = \psi \in Hom(M,M^*)$ and so from (ii),

$$\mu(m) \leq \nu(\eta(\varphi(m))), m \in M$$

$$\therefore \quad \mu(m) \leq \nu(\varphi(m)), m \in M \text{ and } \varphi \in Hom(M,N) \quad (1)$$

Since $\varphi(m) \in N$, we have $\nu(\varphi(m)) \leq \nu'(\varphi(m))$. Then (1) becomes

$$\mu(m) \leq \nu'(\varphi(m)), \forall m \in M \text{ and } \varphi \in Hom(M,N).$$

Therefore μ is ν'-projective. □

3.6. Remark. If μ is any fuzzy G-module on a G-module M, then for $r\in[0,1]$, μ_r: M $\to$ [0,1] defined by

$$\mu_r(m) = r \bullet \mu(m), \forall m \in M$$

is a fuzzy G-module on M and μ exceeds μ_r, for all $r\in[0,1]$. □

3.7. Proposition. *Let μ and ν be the fuzzy G-modules on the G-modules M and M* respectively. If μ is ν_r-projective ($\forall r \in [0,1]$) then μ is ν-projective and the converse hold if ν_r exceeds μ.*

Proof. ($\Rightarrow$) Assume μ is ν_r -projective. Then (i) M is rM* = M*-projective. and (ii) $\mu(m) \leq \nu_r(\psi(m))$, $m \in M$ and $\psi \in$ Hom(M,M*). Since ν exceeds ν_r, for all $r\in[0,1]$, from (ii) we have $\mu(m) \leq \nu(m)$, $\forall m\in M$ and $\psi\in$Hom(M,M*). Therefore μ is ν-projective.

($\Leftarrow$) Assume μ is ν-projective. We have ν exceeds ν_r, for all $r\in[0,1]$, and also it is given that ν_r exceeds μ, so ν exceeds μ. Therefore $\mu(m)\leq\nu(\psi(m))$, $\forall m\in M$ and $\psi\in$Hom(M,M*) . Therefore μ is ν_r-projective.□

3.8.Definition. A fuzzy G-module μ on a G-module M is ***quasi-projective*** if μ is μ-projective.

3.9. Example. Let G = {1,-1} and M = R. Then M is a G-module over R . Also, M is M-projective.

Define $\mu : M \to [0,1]$ by

$$\mu(m) = 1, \quad \text{if } m = 0$$
$$= \tfrac{1}{2}, \quad \text{if } m \neq 0$$

Then μ is a fuzzy G-module on M and $\mu(m) \leq \mu(\psi(m))$, for all $\psi\in$Hom(M,M). Therefore μ is quasi-projective. □

4. Some More Theorems

4.1.Theorem. *Let $M = \overset{n}{\underset{1}{\oplus}} M_i$, where M_i's are G-submodules of the G-module M. Let μ be a fuzzy G-module on M and ν_i's be fuzzy G-modules on M_i*

such that $v = \overset{n}{\underset{1}{\oplus}} v_i$. *Then* μ *is* v *–projective if and only if* μ *is* v_i*-projective, for all i.*

Proof. ($\Rightarrow$) Assume μ is v-projective. Then (i) M is M = M_i-projective. and (ii) $\mu(m) \le v(\psi(m))$, $\forall \psi \in Hom(M,M)$. From (i) and proposition 2.12, we have M is M_i-projective for all i. Let $\varphi \in Hom(M,M_i)$ and $\eta: M_i \to M$ be the inclusion homomorphism. Then $\psi = \eta \bullet \varphi : M \to M$ is a homomorphism and from (ii),

$$\mu(m) \le v(\psi(m)) = v(\eta(\varphi(m)) = v(\varphi(m)), \forall m \in M \quad (1)$$

Since $\varphi \in Hom(M,M_i)$, $\varphi(m) \in M_i$, and

$$\varphi(m) = 0+0+ \ldots\ldots\ldots + \varphi(m) + \ldots\ldots\ldots\ldots +0$$

$$\therefore \quad v(\varphi(m)) = v_1(0) \wedge \ldots\ldots\ldots \wedge v_i(\varphi(m))) \wedge \ldots \wedge v_n(0)$$

$$= v_i(\varphi(m))$$

Hence (1) implies $\mu(m) \le v_i(\varphi(m))$, $\forall m \in M$ and $\varphi \in Hom(M,M_i)$. Therefore μ is v_i-projective for all i.

($\Leftarrow$) Assume μ is v_i-projective for all i. Then (a) M is M_i-projective and (b) $\mu(m) \le v_i(\varphi(m))$ for all $\varphi \in Hom(M,M_i)$. From (a) and from the proposition 2.12, M is $M = \overset{n}{\underset{1}{\oplus}} M_i$-projective. Let $\psi \in Hom(M,M)$. Then $\psi(m) \in M$, $\forall m \in M$. So $\psi(m) = m_1 + m_2 + \ldots\ldots + m_n$, $m_i \in Mi$. Let $\pi_i : M \to M_i$ be the projection map $(1 \le i \le n)$, then $\pi_i(\psi(m)) = m_i$,for all i. Then

$$\psi(m) = \pi_1(m)) + \pi_2(\psi(m)) + \ldots\ldots\ldots + \pi_n(\psi(m))$$

Let $\varphi i = \pi_i \bullet \psi$. Then $\varphi_i \in Hom(M,M_i)$. Then

$$\psi(m) = \varphi_1(m) + \varphi_2(m) + \ldots + \varphi_n(m)$$

From (b), $\mu(m) \le v_i(\varphi_i(m))$, $m \in M$ and for all i. $\le \overset{n}{\underset{1}{\wedge}} \{v_i(\varphi_i(m))\}$

From (2), $v(\psi(m)) = \overset{n}{\underset{1}{\wedge}} \{v_i(\varphi_i(m))\}$

Therefore $\mu(m) \le v(\psi(m))$, $m \in M$ and $\psi \in Hom(M,M)$; and hence μ is v-projective. □

4.2.Corollory. *Let* $M = \bigoplus_{1}^{n} M_i$, *where Mi's are G-submodules of the G-module M. Let* v_i*'s are fuzzy G-modules on* M_i *such that* $v = \bigoplus_{1}^{n} v_i$. *Then v is quasi-projective if and only if v is* v_i*-projective, for all i.*

Proof. Obtained by replacing μ in theorem 4.1 by v. □

We proceed to give below some results regarding fuzzy projectivity; omitting their proofs.

4.3. Theorem. *Let* M_i*'s are fuzzy G-modules. Then the direct sum* $\bigoplus_{1}^{n} M_i$ *is quasi-projective if and only if* M_i *is* M_j*-projective* $(1 \le i, j \le n)$. □

4.4.Corollory. *Let M be a G-module .For a positive integer n,* $M^n = M \oplus M \oplus \ldots \oplus M$ *(n copies) is quasi-projective if and only if M is quasi-projective.* □

4.5. Theorem. *Let* $M = M_1 \oplus M_2$, *where* M_1 *and* M_2 *are G-submodules of M. Let* v_i*'s are fuzzy G-modules on* M_i $(1 \le i \le 2)$ *such that* $v = v_1 \oplus v_2$. *Then v is quasi-projective if and only if* v_i *is* v_j*-projective* $\forall i,j \in \{1,2\}$. □

4.6.Theorem. *Let* $M = \bigoplus_{1}^{n} M_i$, *where* M_i*'s are G-submodules of the G-module M. Let* v_i*'s be fuzzy G-modules on* M_i $(1 \le i \le n)$ *such that* $v = \bigoplus_{1}^{n} v_i$ *Then v is quasi-projective if and only if* v_i *is* v_j*-projective,* $\forall i,j \in \{1,2,\ldots,n\}$□

4.7.Theorem. *Let* $M = \bigoplus_{1}^{n} M_i$ *and* $M^* = \bigoplus_{1}^{n} N_j$, *be G-modules, where* M_i*'s are G-submodules of M and* N_j*'s are G-submodules of M*. Then both M and M* are relatively projective and relatively injective. If μ and v be fuzzy G-modules on M and M* respectively, then μ is v-injective if and only if v is μ-projective.* □

4.8.Theorem. *Let* $M = \bigoplus_{1}^{n} M_i$ *,where* M_i*'s are G-submodules of M . Then M is quasi-injective and quasi-projective. If μ is any fuzzy G-module on M, then μ is quasi-injective if and only if μ is quasi-projective.* □

4.9.Corollory. *Any finite dimensional G-module has a fuzzy G-module* *which is both quasi injective and quasi-projective.* □

4.10.Example. Let $G=(Z_p,x_p)$,where p is prime, be the group of multiplication modulo p. Consider the field $F = (Z_p, +_p, x_p)$. Let $M = F(\sqrt{2}) = \{ a+b\sqrt{2} \ / \ a,b \in F\}$, where + denote $+_p$ (addition modulo p).Then M is a vector space over F. Let $g \in G$ and $m= a+b\sqrt{2} \in M$. Define $g.m = g.(a+b\sqrt{2}) = (g \ x_p \ a) + (g \ x_p \ b) \sqrt{2}$, where + denote $+_p$. Then $gm \in M$ and satisfies

(i) $g.(m_1+m_2) = gm_1+gm_2$

(ii) $(g g')m = g(g' (m))$

(iii) $1.m = m$, for all $m,m_1,m_2 \in M$ & $g,g' \in G$

Therefore M is a G-module . Let $M^* = F^2 = \{ (a,b) / a,b \in F\}$. Let $g \in G$, $m^* = (a,b) \in M^*$. Define $g.m^* = g.(a,b) = (g \ x_p \ a, g \ x_p \ b) \in M^*$. Then M^* also is a G-module . Also $M = F.1 \oplus F\sqrt{2}$ and $M^* = F\varepsilon_1 \oplus F\varepsilon_2$, where $\varepsilon_1=(1,0)$ and $\varepsilon_2=(0,1)$. Then we can show that M and M^* are relatively projective and relatively injective. Define $\mu : M \rightarrow [0, 1]$ by

$$\mu (x) = 1, \text{ if } x = a+b\sqrt{2} =0$$
$$= 3/4, \text{ if } b=0 \ \& \ a \neq 0$$
$$= 1/2 , \text{ if } b \neq 0$$

Then μ is a fuzzy G-module on M. Define $v: M^* \rightarrow [0,1]$ by

$$v (m^*) = 1/2, \text{ if } m^* = 0,$$
$$= 1/4 , \text{ if } m^* \neq 0$$

Then v is a fuzzy G-module on M^*. Also $v (m^*) \leq \mu(1) \wedge \mu (\sqrt{2})$ for all $m^* \in M^*$. Hence by proposition 3.5, v is μ- projective and so by theorem 4.7, μ is v -injective. □

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